A common divisor graph for skew braces

Joint work with Arne Van Antwerpen
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NOTATIONS

A skew brace is a triple $(A, +, \circ)$, where $(A, +)$ and $(A, \circ)$ are groups and

$$a \circ (b + c) = a \circ b - a + a \circ c.$$

$(A, +)$ is the additive group and $(A, \circ)$ is the multiplicative group.

The inverse of a in $(A, +)$ is $-a$ and the inverse of a in $(A, \circ)$ is a'.

Examples: Let $(G, \cdot)$ be a group.

- The trivial skew brace on G is $(G, \cdot, \cdot)$.
- The almost trivial skew brace on G is $(G, \cdot^{op}, \cdot)$.

λ-ACTION

The λ-action of a skew brace $(A, +, \circ)$ is

$$\lambda : (A, \circ) \to \mathrm{Aut}(A, +) \qquad \lambda_a : b \mapsto -a + a \circ b.$$

- skew left distributivity: $a \circ (b + c) = a \circ b + \lambda_a(c)$.
- $I \subseteq A$ is an ideal if:
  $(I, +) \trianglelefteq (A, +)$, $\quad (I, \circ) \trianglelefteq (A, \circ) \quad$ and $\quad \lambda_a(I) = I$ for all $a \in A$.
- The map $r_A \colon A \times A \to A \times A$ defined by

$$(a, b) \mapsto (\lambda_a(b), \lambda_a(b)' \circ a \circ b)$$

  is a set-theoretic solution to the Yang-Baxter equation.

λ-ACTION

For $b \in A$, the λ-orbit of b is

$$\Lambda(b) = \{\lambda_a(b) \colon a \in A\}.$$

The union of the trivial λ-orbits is an additive subgroup:

$$\mathrm{Fix}(A) = \{b \in A \colon \lambda_a(b) = b \quad \forall a \in A\}.$$

Examples: Let $(G, \cdot)$ be a group.

- Trivial skew brace $(G, \cdot, \cdot)$:

$$\lambda_g(h) = g^{-1} \cdot g \cdot h = h.$$

- Almost trivial skew brace $(G, \cdot^{\mathrm{op}}, \cdot)$:

$$\lambda_g(h) = g^{-1} \cdot^{\mathrm{op}} (g \cdot h) = g \cdot h \cdot g^{-1}.$$

DEFINITION

Definition

For a finite skew brace A, let $\Gamma(A)$ be the graph with vertices the non-trivial λ-orbits of A where two vertices C_1, C_2 are adjacent if $\gcd(|C_1|, |C_2|) \neq 1$.

[Bertram–Herzog–Mann] If $(G, \cdot)$ is a finite group, $\Gamma(G)$ is the graph with vertices the non-trivial conjugacy classes of G where two vertices C_1, C_2 are adjacent if $\gcd(|C_1|, |C_2|) \neq 1$.

Connection:

$\Gamma(G, \cdot^{\mathrm{op}}, \cdot) = \Gamma(G)$: on the skew brace $(G, \cdot^{\mathrm{op}}, \cdot)$, the λ-action is

$$\lambda_g(h) = g \cdot h \cdot g^{-1}.$$

EXAMPLES

Let $(A, +, \circ)$ be a finite skew brace.

- $\Gamma(A)$ has no vertices if and only if $+ = \circ$.
- If $|A| = p^2$, then $\Gamma(A)$ is empty or a complete graph with $p - 1$ vertices.
  [Complete classification by Bachiller.]
- If $|A| = pq$, then $\Gamma(A)$ is completely determined by $|\mathrm{Fix}(A)|$.
  [Complete classification by Acri–Bonatto.]

EXAMPLE: SIZE 6

$(A,+)$	$(n,m)\circ(s,t)$	$\lvert\mathrm{Fix}(A)\rvert$	$\Gamma(A)$
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(n+s, m+t)$	6	
$\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$	$(n+(-1)^m s, m+t)$	6	
$\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$	$((-1)^t n+(-1)^m s, m+t)$	3	$\bullet$
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(n+(-1)^m s, m+t)$	2	$\bullet - \bullet$
$\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$	$(n+s, m+t)$	2	$\bullet - \bullet$
$\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$	$((-1)^t n+s, m+t)$	1	$\bullet \quad \bullet$

Table: Skew braces of size 6.

PROPERTIES

Proposition

If A is a finite skew brace such that $\Gamma(A)$ is connected, the diameter of $\Gamma(A)$ is

$$d(\Gamma(A)) \leq 4.$$

Proposition

If A is a finite skew brace, the number of connected components of $\Gamma(A)$ is

$$n(\Gamma(A)) \leq 2.$$

TWO DISCONNECTED VERTICES

Theorem

Let A be a finite skew brace. If $\Gamma(A)$ has exactly two disconnected vertices, then $A \cong (\mathcal{S}_3, \cdot^{\mathrm{op}}, \cdot)$.

$(A,+)$	$(n,m)\circ(s,t)$	$\lvert\mathrm{Fix}(A)\rvert$	$\Gamma(A)$
$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$(n+s,m+t)$	6	
$\mathbb{Z}/3\mathbb{Z}\rtimes_{-1}\mathbb{Z}/2\mathbb{Z}$	$(n+(-1)^m s,m+t)$	6	
$\mathbb{Z}/3\mathbb{Z}\rtimes_{-1}\mathbb{Z}/2\mathbb{Z}$	$((-1)^t n+(-1)^m s,m+t)$	3	•
$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$(n+(-1)^m s,m+t)$	2	•—•
$\mathbb{Z}/3\mathbb{Z}\rtimes_{-1}\mathbb{Z}/2\mathbb{Z}$	$(n+s,m+t)$	2	•—•
$\mathbb{Z}/3\mathbb{Z}\rtimes_{-1}\mathbb{Z}/2\mathbb{Z}$	$((-1)^t n+s,m+t)$	1	• •

Table: Skew braces of size 6.

ONE VERTEX

Theorem

Let A be a skew brace of size $n = 2^m d$, for $\gcd(2, d) = 1$. If $\Gamma(A)$ has exactly one vertex, then $(A, +) \cong F \rtimes \mathbb{Z}/2\mathbb{Z}$ and there exists an abelian group G of odd order such that

$$F = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times G \quad \text{or} \quad F = \mathbb{Z}/2^{m-1}\mathbb{Z} \times G.$$

The number of isomorphism classes of skew braces A with one-vertex graph $\Gamma(A)$ is

$$\begin{cases} m\,\mathrm{Ab}(d) & \text{if } 0 \le m \le 3, \\ 2\,\mathrm{Ab}(d) & \text{if } m \ge 4, \end{cases}$$

$\mathrm{Ab}(d)$ = number of abelian groups of order d [OEIS: A001055].

QUESTIONS

- Applications to solution to the YBE?
- Can we characterize skew braces with a graph with two connected components?
  (Group analog: quasi-Frobenius with abelian kernel and complement [Bertram–Herzog–Mann].)
- Is it true that in the connected case, $d(\Gamma(A)) \leq 3$?
  (For groups [Chillag–Herzog–Mann].)
- When is $d(\Gamma(A)) \leq 2$?

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